THE QUINTIC NONLINEAR SCHRÖDINGER EQUATION ON THREE-DIMENSIONAL ZOLL MANIFOLDS

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Abstract. Let (M, g) be a three-dimensional smooth compact Riemannian manifold such that all geodesics are simple and closed with a common minimal period, such as the 3-sphere $\mathbb{S}^3$ with canonical metric. In this work the global well-posedness problem for the quintic nonlinear Schrödinger equation $i\partial_t u + \Delta u = \pm|u|^4 u$, $u|_{t=0} = u_0$ is solved for small initial data u_0 in the energy space $H^1(M)$, which is the scaling-critical space. Further, local well-posedness for large data, as well as persistence of higher initial Sobolev regularity is obtained. This extends previous results of Burq-Gérard-Tzvetkov to the endpoint case.

1. Introduction and main result

Let (M, g) be a smooth Riemannian manifold without boundary and let $\Delta = \Delta_g$ denote the (negative) Laplace-Beltrami operator on M. Consider the Cauchy problem

$$\begin{aligned} i\partial_t u + \Delta u =& \pm |u|^4 u \\ u|_{t=0} =& \phi \in H^s(M) \end{aligned} \tag{1}$$

If $u : (-T, T) \times M \to \mathbb{C}$ is a sufficiently nice solution to (1) one easily verifies conservation of mass and energy

$$\mathbf{m}(u(t)) = \tfrac{1}{2} \int_M |u(t, x)|^2 dx = \mathbf{m}(\phi), \tag{2}$$

$$\mathbf{e}(u(t)) = \tfrac{1}{2} \int_M |\nabla u(t, x)|^2 \pm \tfrac{1}{3}|u(t, x)|^6 dx = \mathbf{e}(\phi). \tag{3}$$

Therefore, the Sobolev space $H^1(M)$ is the natural energy space for (1), in which for small initial data the local and the global well-posedness problem are at the same level of difficulty. Also, in the three-dimensional Euclidean case $(\mathbb{R}^3, \delta_{ij})$ the scaling

$$u(t, x) \to \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x) \qquad (\lambda > 0)$$

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maps solutions onto solutions and does not alter the $\dot{H}^1(\mathbb{R}^3)$-norm. Therefore, the energy space is called (scaling-)critical.

In continuation of the line of research initiated by Burq-Gérard-Tzvetkov in [4, 6, 5, 7] we focus on three-dimensional Zoll manifolds such as the sphere $M = \mathbb{S}^3$ with canonical metric g, see [1] for more information on the geometric assumption, and (4) below. The sub-quintic problem on $\mathbb{S}^3$ is solved in [6, Theorem 1], and it is proved that the super-quintic problem is ill-posed in [6, Appendix A]. Well-posedness in $H^1(M)$ for the quintic nonlinear Schrödinger equation in $\mathbb{S}^3$ is formulated as an open problem in [6, p. 257, l. 11], and it is shown in [7] that the second Picard-iteration is bounded. This is the starting point for the present paper, in which we prove

Theorem 1.1. *Let $s \geq 1$ and (M, g) be a three-dimensional smooth compact Riemannian manifold such that all geodesics are simple and closed with a common minimal period. Then, the initial value problem* (1) *is locally well-posed in $H^s(M)$, and globally well-posed in $H^s(M)$ if the data is small in $H^1(M)$.*

We refer the reader to Theorem 4.1 in Section 4 for a more precise statement of the main result. Theorem 1.1 completes the small data well-posedness theory for $3d$ Zoll manifolds as we push it to the critical space $H^1(M)$, and Thomann's work [19] shows that the problem is ill-posed in $H^s(M)$ for $s < 1$ and analytic (M, g).

Zoll manifolds have the property that the spectrum of the Laplace-Beltrami operator Δ is clustered around a sequence of squares [9, 8, 21], see (4) below. On the other hand, the spectral cluster estimates [6] are optimal on spheres. This constitutes a sharp contrast to the case of the flat rational torus $M = \mathbb{T}^3$ where we have recently established the analogous result to Theorem 1.1, see [14].

The choice of Zoll manifolds as our setup is motivated from [4, 6, 5]. Also, one of our main ingredients in the proof – the trilinear spectral cluster estimates (Lemma 3.2) – are provided in [6]. In this paper we use critical function space techniques which have been introduced by Tataru and Koch-Tataru [15], see also [10] for further details. They have already been applied to related problems, namely energy-critical Schrödinger equations on $M = \mathbb{T}^3$ [14] and $M = \mathbb{R}^2 \times \mathbb{T}^2, \mathbb{R}^3 \times \mathbb{T}$ [13]. On a technical level, however, we face different challenges in this paper as the estimates for space and time variables will be decoupled, and Galilean invariance and fine orthogonality arguments in the spatial frequencies are unavailable. Actually, our function spaces are slightly different and the general strategy of proof of Corollary 3.7 also provides an alternate approach to [14].

The paper is organized as follows: In Section 2 we describe the geometric and functional setup. Section 3 starts by collecting known estimates on exponential sums and spectral projectors, and after some preparation it concludes with the key estimate of this work in Corollary 3.7. Section 4 contains the main nonlinear estimate and a precise statement of the main result in Theorem 4.1. In the Appendix we describe the necessary modifications with respect to Bourgain's paper [2] in order to conclude Lemma 3.1.

2. Notation and function spaces

Since M is compact, the spectrum $\sigma(-\Delta)$ of $-\Delta = -\Delta_g$ is discrete and we list the nonnegative eigenvalues $0 = \lambda_0^2 \leq \lambda_1^2 \leq \ldots \leq \lambda_n^2 \to +\infty$. Define $h_k : L^2(M) \to L^2(M)$ to be the spectral projector onto the eigenspace E_k corresponding to the eigenvalue λ_k^2. We have the orthogonal decomposition

$$L^2(M) = \bigoplus_{k=0}^{\infty} E_k,$$

Following [4, 6, 5] we assume that (M, g) is a three-dimensional Zoll manifold, i.e. all geodesics are simple and closed with a common minimal period T, and without loss we may assume that $T = 2\pi$. In fact, we are using this assumption only to conclude that the spectrum is clustered around the sequence μ_n^2 where $\mu_n := (n + \alpha/4)$ (for convenience of notation we define $\mu_0 := 0$). More precisely, there exist $\alpha, E \in \mathbb{N}$ such that

$$\sigma(-\Delta) \subset \cup_{n=1}^{\infty} I_n, \text{ where } I_n = [\mu_n^2 - E, \mu_n^2 + E], \tag{4}$$

see [9, 8, 21]. By adding a bounded interval I_0, and increasing α and relabeling if necessary, we may assume without loss of generality that $\sigma(-\Delta) \subset \cup_{n=0}^{\infty} I_n$, where

$$I_n = [\mu_n^2 - E, \mu_n^2 + E] \text{ for } n \geq 1, \text{ and } I_0 = [-B, B], \tag{5}$$

with the additional property that all these intervals are pairwise disjoint. Let $p_n = \sum_{k \in \mathbb{N}_0 : \lambda_k^2 \in I_n} h_k$ for $n \in \mathbb{N}_0$. Note that this is a spectral projector to which the result of [6, Theorem 3] applies. We have

$$\sum_{n=0}^{\infty} p_n = \mathrm{Id}.$$

For subsets $J \subset \mathbb{R}$ we define

$$P_J = \sum_{n \in J \cap \mathbb{N}_0} p_n.$$

Specifically, for dyadic numbers $N = 1, 2, 4, \ldots$ we write

$$P_N = P_{[N,2N)}, \quad P_0 = p_0, \text{ such that } \sum_{N \geq 0} P_N = \mathrm{Id},$$

where we add up all dyadic $N \geq 1$ and $N = 0$.

We define $H^s(M) = (1-\Delta_g)^{-s/2} L^2(M)$ with

$$\|f\|_{H^s(M)}^2 = \sum_{j=0}^{\infty} \langle \lambda_j \rangle^{2s} \|h_j f\|_{L^2(M)}^2,$$

where $\langle x \rangle = (1+|x|^2)^{1/2}$ and observe that

$$\|f\|_{H^s(M)}^2 \sim \sum_{n=0}^{\infty} \langle n \rangle^{2s} \|p_n f\|_{L^2(M)}^2 \sim \sum_{N \geq 0} \langle N \rangle^{2s} \|P_N f\|_{L^2(M)}^2.$$

Remark 1. In the case $M = \mathbb{S}^3$ the eigenfunctions in $E_k = h_k L^2(\mathbb{S}^3)$ are precisely the spherical harmonics of degree k, with eigenvalues

$$\lambda_k^2 = (k+1)^2 - 1,$$

and $\dim(E_k) = (k+1)^2$, see e.g. [18, Section 8.4]. With $E = 1$, $\alpha = 4$ in (4) we have $\lambda_k^2 \in I_n$ if and only if $k = n$, and $\lambda_n^2 = \mu_n^2 - 1$ for $n \geq 2$.

For technical purposes we introduce the operator $\widetilde{\Delta}$ defined by

$$-\widetilde{\Delta}\phi = \sum_{n=1}^{\infty} \mu_n^2 p_n \phi,$$

which is similar to the construction in [5, formula (3.5)].

Let us quickly review the theory of the critical function spaces U^p and V^p which have been introduced in the context of dispersive PDEs by Tataru and Koch-Tataru, see [15]. We refer the reader to the papers [10, 14] for more details. Let $\chi_I : \mathbb{R} \to \mathbb{R}$ denote the sharp characteristic function of a set $I \subset \mathbb{R}$. Let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K \leq \infty$ of the real line. If $t_K = \infty$, we use the convention that $v(t_K) := 0$ for all functions $v : \mathbb{R} \to L^2$.

Definition 2.1. Let $1 \leq p < \infty$.

(i) Any step-function $a : \mathbb{R} \to L^2$,

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)} \phi_{k-1}$$

with $\{t_k\} \in \mathcal{Z}$, $\{\phi_k\} \subset L^2$ s.t. $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$, is called a U^p-atom. We define U^p as the corresponding atomic space, i.e.

the space of all $u : \mathbb{R} \to L^2$ which can be written as

$$u = \sum_{j=1}^{\infty} \alpha_j a_j \tag{6}$$

with U^p-atoms a_j, and $\{\alpha_j\} \in \ell^1(\mathbb{N}, \mathbb{C})$. The norm of a function $u \in U^p$ is defined as $\inf \sum_{j=1}^{\infty} |\alpha_j|$, where the infimum is taken over all atomic representations (6) of u.

(ii) Define V^p as the space of all right-continuous functions $v : \mathbb{R} \to L^2$ s.t.

$$\|v\|_{V^p} := \sup_{\{t_k\} \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}}, \tag{7}$$

is finite (here we use the convention $v(\infty) = 0$), and additionally satisfying $\lim_{t \to -\infty} v(t) = 0$.

(iii) If $L^2 = L^2(M; \mathbb{C})$, and A denotes either the standard Laplacian Δ or $\widetilde{\Delta}$, we also define $U_A^p = e^{itA} U^p$ and $V_A^p = e^{itA} V^p$.

Remark 2. (i) Note that the space V^p corresponds to $V^p_{-,rc}$ in [10].

(ii) The spaces U^p, V^p and U_A^p, V_A^p are Banach spaces of bounded functions which are right-continuous and tend to 0 as $t \to -\infty$.

(iii) For $1 \le p < q < \infty$ it holds

$$U_A^p \hookrightarrow V_A^p \hookrightarrow U_A^q \hookrightarrow L^\infty(\mathbb{R}; L^2).$$

Our aim is to control the evolution up to time $T \sim 1$. But on bounded time intervals the flows associated to the operators Δ and $\widetilde{\Delta}$ stay close, a statement which is made precise next.

Lemma 2.2. *Let $1 \le p < \infty$ and τ be a bounded time interval, and let $u : \mathbb{R} \to L^2(M; \mathbb{C})$ be supported in τ. Then, $u \in U_\Delta^p$ if and only if $u \in U_{\widetilde{\Delta}}^p$, with equivalent norms.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R})$ be a smooth cutoff function which is constantly equal to 1 on τ. The claim follows from the fact that

$$\psi e^{\pm it(\Delta - \widetilde{\Delta})} : U^p \to U^p$$

are bounded operators. It suffices to consider an atom a. With $B_j := \sum_{n=0}^{\infty} \sum_{k \in \mathbb{N}_0 : \lambda_k^2 \in I_n} (\pm\lambda_k^2 \mp \mu_n^2)^j h_k$ we write

$$\psi(t) e^{\pm it(\Delta - \widetilde{\Delta})} a(t) = \sum_{j=0}^{\infty} \frac{\psi(t)(it)^j}{j!} B_j a(t),$$

In view of (5) we have $\|B_j\|_{L^2 \to L^2} \le b^j$, where $b = \max\{E, B\} \ge 1$. This implies $B_j a \in U^p$ with $\|B_j a\|_{U^p} \le b^j$. Also, multiplication by ψ, as

well as multiplication by t on the support of ψ are bounded operations in U^p, which follows by duality [10, Remark 2.11]. This implies

$$\|t^j \psi B_j a\|_{U^p} \leq c^j.$$

The claim follows, since $\sum_{j=0}^{\infty} \frac{c^j}{j!} < +\infty$ and U^p is a complete space. □

Definition 2.3. Let $s \in \mathbb{R}$.

(i) We define X^s as the space of all functions $u : \mathbb{R} \to H^s(M; \mathbb{C})$ such that the maps $P_N(u(\cdot)) : \mathbb{R} \to L^2(M; \mathbb{C})$ are in U^2_Δ for all dyadic $N \geq 0$, endowed with the norm

$$\|u\|_{X^s} := \left(\sum_{N \geq 0} \langle N \rangle^{2s} \|P_N u\|^2_{U^2_\Delta} \right)^{\frac{1}{2}}. \tag{8}$$

(ii) We define Y^s as the space of all functions $u : \mathbb{R} \to H^s(M; \mathbb{C})$ such that the maps $P_N(u(\cdot)) : \mathbb{R} \to L^2(M; \mathbb{C})$ are in V^2_Δ for all dyadic $N \geq 0$, equipped with the norm

$$\|u\|_{Y^s} := \left(\sum_{N \geq 0} \langle N \rangle^{2s} \|P_N u\|^2_{V^2_\Delta} \right)^{\frac{1}{2}}. \tag{9}$$

For a time interval $\tau \subset \mathbb{R}$ we define $X^s(\tau)$ and $Y^s(\tau)$ to be the corresponding restriction space.

The above remark implies

$$U^2_\Delta \hookrightarrow X^0 \hookrightarrow Y^0 \hookrightarrow V^2_\Delta.$$

Moreover, there is a useful interpolation type property of U^p and V^p spaces, see [10, Proposition 2.20] and [14, Lemma 2.4].

Lemma 2.4. *Let $q_1, q_2, q_3 > 2$, τ be a time interval, and*

$$T : U^{q_1}_\Delta \times U^{q_2}_\Delta \times U^{q_3}_\Delta \to L^2(\tau \times M)$$

be a bounded, tri-linear operator with $\|T(u_1, u_2, u_3)\|_{L^2} \leq C \prod_{j=1}^3 \|u_j\|_{U^{q_j}_\Delta}$. In addition, assume that there exists $C_2 \in (0, C]$ such that the estimate $\|T(u_1, u_2, u_3)\|_{L^2} \leq C_2 \prod_{j=1}^3 \|u_j\|_{U^2_\Delta}$ holds true. Then, T satisfies the estimate

$$\|T(u_1, u_2, u_3)\|_{L^2} \lesssim C_2 (\ln \frac{C}{C_2} + 1)^3 \prod_{j=1}^3 \|u_j\|_{V^2_\Delta}, \quad u_j \in V^2_\Delta, \ j = 1, 2, 3.$$

Let $\tau = [a, b)$, $f \in L^1(\tau; L^2(M; \mathbb{C}))$ and define

$$\mathcal{I}(f)(t) := \int_a^t e^{i(t-s)\Delta} f(s) ds \tag{10}$$

for $t \in \tau$ and $\mathcal{I}(f)(t) = 0$ for $t < a$ and $\mathcal{I}(f)(t) = \mathcal{I}(f)(b)$ for $t \geq b$.

Lemma 2.5. *Let $s \in \mathbb{R}$, $\tau = [a, b) \subset \mathbb{R}$.*

(i) *For all $u_0 \in H^s(M)$ and $u(t) := \chi_\tau(t)e^{it\Delta}u_0$ we have $u \in X^s(\tau)$ and*
$$\|u\|_{X^s(\tau)} \lesssim \|u_0\|_{H^s}. \tag{11}$$

(ii) *Let $P_N f \in L^1(\tau; L^2(M))$ for all $N \geq 0$. Then, $\sum_{N\geq 0} \mathcal{I}(P_N f) =: \mathcal{I}(f)$ converges in $X^s(\tau)$ and*
$$\|\mathcal{I}(f)\|_{X^s(\tau)} \leq \sup_{\|v\|_{Y^{-s}(\tau)}=1} \Big| \sum_{N\geq 0} \int_\tau \int_M P_N f(t,x)\overline{v(t,x)}dxdt \Big|, \tag{12}$$
provided that the r.h.s in (12) is finite.

We refer the reader to [14, Propositions 2.10 and 2.11] and [10, Propositions 2.8 and 2.10] for analogous statements and proofs, which apply here with trivial modifications.

3. Linear and multilinear estimates

We use an extension of Bourgain's estimate [2, Proposition 1.10 and Section 4] on exponential sums which is due to Burq–Gérard–Tzvetkov [7, Lemma 5.3] in the case $p = 6$, $\mu_n = n$. For convenience we choose $\tau_0 = [0, 32\pi]$ as our base time interval, as this is a joint period of $e^{-it\mu_n^2}$.

Lemma 3.1. *Let $p > 4$ and $\alpha \in \mathbb{N}_0$. It holds that*
$$\Big\| \sum_{n\in\mathbb{Z}\cap J} c_n e^{-it\mu_n^2} \Big\|_{L_t^p(\tau_0)} \lesssim N^{1/2-2/p} \Big(\sum_{n\in\mathbb{Z}\cap J} |c_n|^2 \Big)^{\frac{1}{2}} \tag{13}$$
for every $J = [b, b+N]$ with $N \geq 1$ and the sequence $\mu_n = n + \alpha/4$.

A proof can be found in Appendix A. We will also rely on the trilinear spectral projector estimate of Burq–Gérard–Tzvetkov [6], which is valid on every smooth Riemannian three-manifold (M, g).

Lemma 3.2 ([6], Theorem 3)**.** *Let $0 < \epsilon \ll 1$. For all integers $n_1 \geq n_2 \geq n_3 \geq 0$ and $f_1, f_2, f_3 \in L^2(M)$ the estimate*
$$\|p_{n_1}f_1 p_{n_2}f_2 p_{n_3}f_3\|_{L^2(M)} \lesssim \langle n_2\rangle^{1/2+\epsilon}\langle n_3\rangle^{1-\epsilon} \prod_{k=1}^{3} \|p_{n_k}f_k\|_{L^2(M)} \tag{14}$$
holds true.

In the nonlinear analysis we need another useful and well-known estimate concerning the spectral localization of products of eigenfunctions.

Lemma 3.3. *If $N_0 \gg N_1, N_2, N_3$ are dyadic and $\gamma \geq 1$, then*

$$\Big| \int_M P_{N_0} f_0 P_{N_1} f_1 P_{N_2} f_2 P_{N_3} f_3 dx \Big| \lesssim N_0^{-\gamma} \prod_{j=0}^{3} \|P_{N_j} f_j\|_{L^2(M)} \tag{15}$$

where the implicit constant depends only on γ.

Proof. For single eigenfunctions it can be found in [5, Lemma 2.6] (written for $d = 2$) or more generally in [11, Section 4]. By the Weyl asymptotic the number of eigenvalues $\lambda_k^2 \in I_{n_j}$ grows at most like n_j^2 in dimension $d = 3$, which implies

$$\begin{aligned}
&\Big| \int_M P_{N_0} f_0 P_{N_1} f_1 P_{N_2} f_2 P_{N_3} f_3 dx \Big| \\
\leq & \sum_{n_j \sim N_j} \sum_{k_j : \lambda_{k_j}^2 \in I_{n_j}^2} \Big| \int_M h_{k_0} f_0 h_{k_1} f_1 h_{k_2} f_2 h_{k_3} f_3 dx \Big| \\
\lesssim & \sum_{n_j \sim N_j} \sum_{k_j : \lambda_{k_j}^2 \in I_{n_j}^2} k_0^{-\gamma-10} \prod_{j=0}^{3} \|h_{k_j} f_j\|_{L^2} \lesssim N_0^{-\gamma} \prod_{j=0}^{3} \|P_{N_j} f_j\|_{L^2},
\end{aligned}$$

by Cauchy-Schwarz, cp. also [5, Lemma 2.7] for similar arguments. □

Note that Lemma 3.3 is trivial in specific cases such as $M = \mathbb{S}^3$ with canonical metric.

For later reference we explicitely state a crude bound which disregards all oscillations in time.

Lemma 3.4. *For all $u_1, u_2, u_3 \in L^\infty(\tau; L^2(M))$, and dyadic $N_1 \geq N_2 \geq N_3 \geq 0$ and time intervals τ the estimate*

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim |\tau|^{\frac{1}{2}} N_2^{\frac{3}{2}} N_3^{\frac{3}{2}} \prod_{j=1}^{3} \|u_j\|_{L^\infty(\tau; L^2(M))} \tag{16}$$

holds true.

Proof. By Hölder's inequality we obtain

$$\begin{aligned}
&\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \\
\leq & |\tau|^{\frac{1}{2}} \|P_{N_1} u_1\|_{L^\infty(\tau; L^2(M))} \|P_{N_2} u_2\|_{L^\infty(\tau \times M)} \|P_{N_3} u_3\|_{L^\infty(\tau \times M)}.
\end{aligned}$$

For t fixed we have

$$\begin{aligned}
\|P_{N_j} u_j(t)\|_{L^\infty(M)} \leq & \sum_{n_j \sim N_j} \|p_{n_j} u_j(t)\|_{L^\infty(M)} \\
\lesssim & \sum_{n_j \sim N_j} n_j \|p_{n_j} u_j(t)\|_{L^2(M)} \lesssim N_j^{\frac{3}{2}} \|P_{N_j} u_j(t)\|_{L^2(M)}
\end{aligned}$$

by Sogge's estimate [16, Proposition 2.1] and the Cauchy-Schwarz inequality. The claim follows by taking the supremum in t. $\square$

Next, we prove (dyadic) Strichartz estimates in a restricted range, generalizing [7, Proposition 5.1].

Lemma 3.5. *Let $p > 4$. Then, for all $N \geq 0$ we have*

$$\|P_N e^{it\Delta}\phi\|_{L^p(\tau_0 \times M)} \lesssim \langle N\rangle^{\frac{3}{2}-\frac{5}{p}}\|\phi\|_{L^2}. \tag{17}$$

and

$$\|P_N u\|_{L^p(\tau_0 \times M)} \lesssim \langle N\rangle^{\frac{3}{2}-\frac{5}{p}}\|u\|_{U^p_\Delta}. \tag{18}$$

Proof. *a)* First, we prove estimate (17) with $\widetilde{\Delta}$ replacing Δ. We write

$$P_N e^{it\widetilde{\Delta}}\phi(x) = \sum_{n\sim N} e^{-it\mu_n^2} p_n\phi(x),$$

and Lemma 3.1 yields

$$\|P_N e^{it\widetilde{\Delta}}\phi(x)\|_{L^p(\tau_0)} \lesssim \langle N\rangle^{\frac{1}{2}-\frac{2}{p}}\Big(\sum_{n\sim N}|p_n\phi(x)|^2\Big)^{\frac{1}{2}}$$

Integration in x, an application of Minkowski's inequality and the dual of Sogge's estimate [16, formula (2.3)] imply that

$$\begin{aligned}\Big\|\Big(\sum_{n\sim N}|p_n\phi(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(M)} &\lesssim \Big(\sum_{n\sim N}\|p_n\phi\|_{L^p}^2\Big)^{\frac{1}{2}}\\ \lesssim \Big(\sum_{n\sim N}\langle n\rangle^{2-\frac{6}{p}}\|p_n\phi\|_{L^2}^2\Big)^{\frac{1}{2}} &\lesssim \langle N\rangle^{1-\frac{3}{p}}\|\phi\|_{L^2},\end{aligned}$$

where we have used orthogonality in the last step.

b) Now, let $u \in U^p_\Delta$. By Lemma 2.2 it suffices to prove the bound (18) for a $U^p_{\widetilde{\Delta}}$-atom

$$u(t,x) = \sum_{k=1}^{K}\chi_{[t_{k-1},t_k)}e^{it\widetilde{\Delta}}\phi_{k-1}(x), \quad \sum_{k=0}^{K-1}\|\phi_k\|_{L^2}^p = 1.$$

Estimate (17) yields

$$\begin{aligned}\|P_N u\|_{L^p(\tau_0\times M)} &\leq \Big(\sum_{k=1}^{K}\|P_N e^{it\widetilde{\Delta}}\phi_{k-1}\|^p_{L^p(\tau_0\times M)}\Big)^{\frac{1}{p}}\\ &\lesssim \langle N\rangle^{\frac{3}{2}-\frac{5}{p}}\Big(\sum_{k=1}^{K}\|P_N\phi_{k-1}\|^p_{L^2(M)}\Big)^{\frac{1}{p}}\\ &\lesssim \langle N\rangle^{\frac{3}{2}-\frac{5}{p}}.\end{aligned}$$

This proves the second bound (18), which in turn implies (17) and the proof is complete. □

E.g. on the sphere $M=\mathbb{S}^3$, the restriction to dyadic frequency bands can be removed by Littlewood-Paley theory [17], but we do not need it here. Also, the loss of derivatives precisely matches the loss on $\mathbb{R}^3$ coming from the sharp Strichartz estimate and the Sobolev embedding, and also Bourgain's bound on $\mathbb{T}^3$ [3].

For an interval J we write $P_{N,J}=P_J P_N$. The next Proposition is an extension and improvement of [7, Theorem 5.1].

Proposition 3.6. *Let $\delta\in(0,\frac{1}{2})$ and $\eta>0$. Then, for all $u_1,u_2,u_3\in U^2_\Delta$, and dyadic $N_1\geq N_2\geq N_3\geq 0$ the estimate*

$$\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)} \lesssim \Big(\frac{\langle N_2\rangle}{\langle N_1\rangle}+\frac{1}{\langle N_2\rangle}\Big)^{\delta}\langle N_2\rangle^{\frac{1}{2}+\eta+\delta}\langle N_3\rangle^{\frac{3}{2}-\eta-\delta}\prod_{j=1}^{3}\|u_j\|_{U^2_\Delta}. \tag{19}$$

holds true.

Proof. a) In the case $N_3\leq N_2\leq 1$ the l.h.s. is bounded by our crude estimate (16) in conjunction with Remark 2 (iii). Henceforth we assume $N_2\geq 1$. Since τ_0 is a compact interval, it suffices to prove the corresponding bound in $U^2_{\widetilde{\Delta}}\times U^2_{\widetilde{\Delta}}\times U^2_{\widetilde{\Delta}}$ by Lemma 2.2. Further, by definition of the spaces it suffices to consider $U^2_{\widetilde{\Delta}}$-atoms

$$u_j(t,x)=\sum_{k_j=1}^{K_j}\chi_{[t^{(j)}_{k_j-1},t^{(j)}_{k_j})}e^{it\Delta}\phi^{(j)}_{k_j-1}(x),\quad \sum_{k_j=0}^{K_j-1}\|\phi^{(j)}_{k_j}\|^2_{L^2}=1,$$

for $j=1,2,3$. However, in this case

$$\|\prod_{j=1}^{3}P_{N_j}u_j\|^2_{L^2(\tau_0\times M)}\leq\sum_{k_1,k_2,k_3}\|\prod_{j=1}^{3}P_{N_j}e^{it\Delta}\phi^{(j)}_{k_j-1}\|^2_{L^2(\tau_0\times M)},$$

so the claim (19) follows if we can show it in the case

$$u_j=e^{it\widetilde{\Delta}}\phi_j,\quad j=1,2,3. \tag{20}$$

b) Assume (20). We define the partition

$$\mathbb{N}_0=\dot\cup_{m\in\mathbb{N}_0}J_m\quad\text{ where } J_m=[mN_2^2/N_1,(m+1)N_2^2/N_1)\cap\mathbb{N}_0$$

in order to proceed similarly to [14, proof of (26), p. 341–342] and [13]: For fixed $x\in M$ it holds

$$\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3(x)\|^2_{L^2(\tau_0)}\sim\sum_m\|P_{N_1,J_m}u_1P_{N_2}u_2P_{N_3}u_3(x)\|^2_{L^2(\tau_0)},$$

due to almost orthogonality induced by the time oscillations. Indeed, it holds that

$$\langle P_{N_1,J_m}u_1P_{N_2}u_2P_{N_3}u_3(x), P_{N_1,J_{m'}}u_1P_{N_2}u_2P_{N_3}u_3(x)\rangle_{L^2(\tau_0)}$$
$$= \sum_{\substack{n_j,{n_j}'\sim N_j\\ n_1\in J_m, {n_1}'\in J_{m'}}} I_{n_1,n_2,n_3}^{{n_1}',{n_2}',{n_3}'} \prod_{j=1}^{3} p_{n_j}\phi_j(x)\overline{p_{{n_j}'}\phi_j(x)}$$

where

$$I_{n_1,n_2,n_3}^{{n_1}',{n_2}',{n_3}'} = \int_{\tau_0} e^{-it\mu_{n_1}^2}e^{-it\mu_{n_2}^2}e^{-it\mu_{n_3}^2}e^{it\mu_{{n_1}'}^2}e^{it\mu_{{n_2}'}^2}e^{it\mu_{{n_3}'}^2}dt.$$

For every $|m-m'|\gg 1$ and $n_1,{n_1}'\sim N_1$ such that $n_1\in J_m, {n_1}'\in J_{m'}$ and all $n_2,{n_2}'\sim N_2$, $n_3,{n_3}'\sim N_3$ we have the following estimate for the phase

$$|\sum_{j=1}^{3}(\mu_{{n_j}'}^2-\mu_{n_j}^2)|\geq|\mu_{{n_1}'}^2-\mu_{n_1}^2|-8N_2^2\gtrsim|m-m'|N_2^2,$$

because $\mu_{{n_1}'}+\mu_{n_1}\geq N_1$, which implies

$$I_{n_1,n_2,n_3}^{{n_1}',{n_2}',{n_3}'}=0.$$

c) By parts a) and b) the claim is reduced to showing that

$$\|P_{N_1,J}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\lesssim|J|^\delta N_2^{\frac{1}{2}+\eta}\langle N_3\rangle^{\frac{3}{2}-\eta-\delta}\prod_{j=1}^{3}\|\phi_j\|_{L^2}. \quad (21)$$

for u_j of the form (20), and intervals of length $|J|\geq 1$. This is a refinement of [7, Theorem 5.1], which is proved as follows: For fixed $x\in M$ we obtain by Hölder's inequality

$$\|P_{N_1,J}u_1P_{N_2}u_2P_{N_3}u_3(x)\|_{L^2(\tau_0)}$$
$$\leq\|P_{N_1,J}u_1(x)\|_{L^{p_1}(\tau_0)}\|P_{N_2}u_2(x)\|_{L^{p_2}(\tau_0)}\|P_{N_3}u_3(x)\|_{L^{p_3}(\tau_0)}$$

where $1/p_1+1/p_2+1/p_3=1/2$ which we choose to satisfy $4<p_1,p_2,p_3<+\infty$. An application of (13) gives

$$\|P_{N_1,J}u_1(x)\|_{L_t^{p_1}(\tau_0)}\lesssim|J|^{\frac{1}{2}-\frac{2}{p_1}}\Big(\sum_{\substack{n_1\sim N_1\\ n_1\in J}}|p_{n_1}\phi_1(x)|^2\Big)^{\frac{1}{2}}$$

and also

$$\|P_{N_j}u_j(x)\|_{L_t^{p_j}(\tau_0)}\lesssim N_j^{\frac{1}{2}-\frac{2}{p_j}}\Big(\sum_{n_j\sim N_j}|p_{n_j}\phi_j(x)|^2\Big)^{\frac{1}{2}}$$

for $j=2,3$. By integration with respect to $x\in M$ we obtain

$$\begin{aligned}
&\|P_{N_1,J}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\\
\lesssim{}& |J|^{\frac12-\frac{2}{p_1}}N_2^{\frac12-\frac{2}{p_2}}N_3^{\frac12-\frac{2}{p_3}}\Big(\sum_{n_j\sim N_j}\|\prod_{j=1}^3 p_{n_j}\phi_j\|_{L^2(M)}^2\Big)^{\frac12}\\
\lesssim{}& |J|^{\frac12-\frac{2}{p_1}}N_2^{1+\epsilon-\frac{2}{p_2}}N_3^{\frac32-\epsilon-\frac{2}{p_3}}\prod_{j=1}^3\|\phi_j\|_{L^2(M)}
\end{aligned}$$

for any small $\epsilon>0$, where we have used the trilinear spectral cluster estimate (14) in the last step, similar to the proof of [7, Theorem 5.1]. The claim follows with $\delta=\frac12-\frac{2}{p_1}\in(0,\frac12)$ by choosing $p_2>4$ and $\epsilon>0$ small enough such that $\epsilon+\frac12-\frac{2}{p_2}=\eta$. $\square$

Finally, we transfer the bound to V^2_Δ by interpolation and obtain a result which corresponds to [14, Propositon 3.5] in the case of $M=\mathbb{T}^3$. The argument, however, is slightly different from the one in [14, 13] as it does not involve finer than dyadic scales.

Corollary 3.7. *There exists $\alpha>0$, such that for all $u_1,u_2,u_3\in V^2_\Delta$, and dyadic $N_1\geq N_2\geq N_3\geq 0$ the estimate*

$$\begin{aligned}
&\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\\
\lesssim{}& \max\Big\{\frac{\langle N_3\rangle}{\langle N_1\rangle},\frac{1}{\langle N_2\rangle}\Big\}^\alpha\langle N_2\rangle\langle N_3\rangle\prod_{j=1}^3\|u_j\|_{V^2_\Delta}.
\end{aligned}\tag{22}$$

holds true.

Proof. We restrict our attention to the nontrivial case $N_1\geq 1$, and treat the two cases

$$\text{a) } N_2^2\geq N_1 \qquad \text{b) } N_2^2<N_1$$

separately.

Case a) For $p,q>4$ satisfying $\frac2p+\frac1q=\frac12$ we exploit (18)

$$\begin{aligned}
&\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\\
\leq{}& \|P_{N_1}u_1\|_{L^p(\tau_0\times M)}\|P_{N_2}u_2\|_{L^p(\tau_0\times M)}\|P_{N_3}u_3\|_{L^q(\tau_0\times M)}\\
\lesssim{}& N_1^{\frac32-\frac5p}N_2^{\frac32-\frac5p}\langle N_3\rangle^{\frac32-\frac5q}\|P_{N_1}u_1\|_{U^p_\Delta}\|P_{N_2}u_2\|_{U^p_\Delta}\|P_{N_3}u_3\|_{U^q_\Delta}.
\end{aligned}$$

Let $\rho>0$ be small. We choose $p>4$ such that $\frac32-\frac5p=\frac14+\rho$. Then,

$$\begin{aligned}
&\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\\
\leq{}& \Big(\frac{N_1}{N_2}\Big)^{\frac14+\rho}N_2^{\frac12+2\rho}\langle N_3\rangle^{\frac32-2\rho}\|P_{N_1}u_1\|_{U^p_\Delta}\|P_{N_2}u_2\|_{U^p_\Delta}\|P_{N_3}u_3\|_{U^q_\Delta}.
\end{aligned}\tag{23}$$

Interpolating (19) and (23) via Lemma 2.4 yields

$$\begin{aligned}&\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\\ &\lesssim \left(\frac{N_2}{N_1}\right)^{\delta'}\langle N_2\rangle^{\frac{1}{2}+2\delta'}\langle N_3\rangle^{\frac{3}{2}-2\delta'}\prod_{j=1}^{3}\|u_j\|_{V^2_\Delta}.\end{aligned}$$

for small $\delta'>0$, because in the present case we have $N_2^2\geq N_1$.

Case b) In this case where $N_2^2<N_1$, the key is to observe that (19) provides the subcritical bound

$$\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\lesssim\langle N_2\rangle^{\frac{1}{2}+\eta}\langle N_3\rangle\prod_{j=1}^{3}\|u_j\|_{U^2_\Delta}. \tag{24}$$

for any $\eta>0$. On the other hand, estimate (16) and Remark 2 (iii) imply

$$\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\lesssim\langle N_2\rangle^{\frac{3}{2}}\langle N_3\rangle^{\frac{3}{2}}\prod_{j=1}^{3}\|u_j\|_{U^p_\Delta}.$$

for any $p\in[1,\infty)$, so interpolation via Lemma 2.4 yields

$$\|P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2(\tau_0\times M)}\lesssim\langle N_2\rangle^{\frac{1}{2}+\eta'}\langle N_3\rangle\prod_{j=1}^{3}\|u_j\|_{V^2_\Delta}. \tag{25}$$

for any $\eta'>0$, which implies (22) in this case. □

4. The main result

As usual, we rewrite the initial value problem as an integral equation

$$u(t)=e^{it\Delta}\phi\mp i\mathcal{I}(|u|^4u)(t). \tag{26}$$

Now, we restate Theorem 1.1 in a more precise form. Let us denote the ball in $H^1(M)$ with center ϕ and radius ε by $B_\varepsilon(\phi)$.

Theorem 4.1. *Let (M,g) be a three-dimensional compact smooth Riemannian manifold with Laplace-Beltrami operator Δ satisfying the spectral condition (4), and let $s\geq 1$.*

(i) *(Local well-posedness) For every $\phi_*\in H^1(M)$ there exists $\varepsilon>0$ and $T=T(\phi_*)>0$ such that the following holds true:*
   (a) *For all initial data $\phi\in B_\varepsilon(\phi_*)\cap H^s(M)$ the Cauchy problem (26) has a unique solution*

$$u=:\Phi(\phi)\in C([0,T);H^s(M))\cap X^s([0,T)).$$

(b) *The solution constructed in Part (i)(a) obeys the conservation laws* (3) *and* (2)*, and the flow map*

$$\Phi : B_\varepsilon(\phi_*) \cap H^s(M) \to C([0,T); H^s(M)) \cap X^s([0,T))$$

is Lipschitz continuous.

(ii) *(Global well-posedness for small data) With $\phi_* = 0$ there exists $\varepsilon_0 > 0$ such that for all $T > 0$ the assertions (i)(a) and (i)(b) above hold true.*

The proof is very similar to the proof of [14, Theorems 1.1 and 1.2], as it is based on the following Proposition 4.2 which corresponds to [14, Proposition 4.1].

Proposition 4.2. *Let $s \geq 1$. Then, for all intervals $\tau \subset \tau_0$ and all $u_j \in X^s(\tau)$, $j = 1, \ldots, 5$, the estimate*

$$\Big\|\mathcal{I}\Big(\prod_{j=1}^{5} \widetilde{u_j}\Big)\Big\|_{X^s(\tau)} \lesssim \sum_{k=1}^{5} \|u_k\|_{X^s(\tau)} \prod_{j=1;j\neq k}^{5} \|u_j\|_{X^1(\tau)} \tag{27}$$

holds true, where $\widetilde{u_j}$ denotes either u_j or its complex conjugate $\overline{u}_j$.

Proof. The proof is a variation of the proof of [14, Proposition 4.1], so we focus on the new aspects here: Lemma 2.5 implies that $\mathcal{I}(\prod_{j=1}^5 \widetilde{u_j}) \in X^s(\tau)$ and

$$\Big\|\mathcal{I}(\prod_{j=1}^{5} \widetilde{u_j})\Big\|_{X^s(\tau)} \leq \sup_{\|u_0\|_{Y^{-s}(\tau)}=1} \Big| \sum_{N_0 \geq 0} \int_\tau \int_M P_{N_0} \prod_{j=1}^{5} \widetilde{u_j}\, \overline{u_0} dxdt \Big|,$$

provided that the r. h. s. is finite. Thus, by choosing suitable extensions (which we also denote by u_j) the claim is reduced to proving

$$\Big| \sum_{N_0 \geq 0} \int_\tau \int_M P_{N_0} \widetilde{u_0} \prod_{j=1}^{5} \widetilde{u_j}\ dxdt \Big| \lesssim \|u_0\|_{Y^{-s}} \sum_{k=1}^{5} \|u_k\|_{X^s} \prod_{k=1;k\neq j}^{5} \|u_k\|_{X^1}, \tag{28}$$

We dyadically decompose each $\widetilde{u_k}$, and by symmetry in $u_1, \ldots, u_5$ it suffices to consider

$$\Sigma := \sum_{N_0 \geq 0;\, N_1 \geq \ldots \geq N_5 \geq 0} \int_\tau \int_M \prod_{j=0}^{5} P_{N_j} \widetilde{u_j}\ dxdt$$

We split the sum $\Sigma = \Sigma_1 + \Sigma_2$, where Σ_1 is defined by the constraint $\max\{N_0, N_2\} \sim N_1$. Σ_1 is the major contribution which can be handled by means of the Cauchy-Schwarz inequality and Corollary 3.7 precisely

as in the proof of [14, Proposition 4.1]. The result is

$$\Sigma_1 \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{X^s} \prod_{j=2}^{5} \|u_j\|_{X^1}.$$

In fact, in specific cases such as $M = \mathbb{S}^3$ there will be no further contribution because the product of five spherical harmonics of maximal degree k can be developed into a series of spherical harmonics of maximal degree $5k$. In general, however, it remains to consider a minor contribution of lower order, which comes from the range where $\max\{N_0, N_2\} \ll N_1$ or $N_1 \ll N_0$, and which we split $\Sigma_2 = \Sigma_{21} + \Sigma_{22}$ accordingly. We have

$$\Sigma_{21} \le \sum_{N_2 \ge \ldots \ge N_5;\ N_1 \gg N_0, N_2} \sum_{L \ge 0} \Big| \int_\tau I(N_0, \ldots, N_5, L)(t) dt \Big|,$$

where

$$I(N_0, \ldots, N_5, L)(t) = \int_M P_L \Big(\prod_{j=0}^{2} P_{N_j} \widetilde{u_j} \Big) \prod_{j=3}^{5} P_{N_j} \widetilde{u_j} dx.$$

If $L \gtrsim N_1$ we apply Lemma 3.3 (recall that $N_3, N_4, N_5 \ll L$), to deduce

$$\begin{aligned} &|I(N_0, \ldots, N_5, L)(t)| \\ \lesssim\ & L^{-5} \|P_L(P_{N_0} u_0 P_{N_1} u_1 P_{N_2} u_2)(t)\|_{L^2(M)} \prod_{j=3}^{5} \|P_{N_j} u_j(t)\|_{L^2(M)}, \end{aligned}$$

and Hölder's inequality and Lemma 3.4 imply

$$\int_\tau |I(N_0, \ldots, N_5, L)(t)| dt \lesssim L^{-5} N_0^{\frac{3}{2}} N_2^{\frac{3}{2}} \prod_{j=0}^{5} \|P_{N_j} u_j\|_{L^\infty(\tau_0; L^2(M))},$$

which implies

$$\sum_{L \gtrsim N_1} \Big| \int_\tau I(N_0, \ldots, N_5, L)(t) dt \Big| \lesssim N_1^{-2} \prod_{j=0}^{5} \|P_{N_j} u_j\|_{V^2_\Delta}. \tag{29}$$

On the other hand, if $L \ll N_1$ we apply Lemma 3.3 (in this case $L, N_0, N_2 \ll N_1$), to deduce

$$\begin{aligned} &|I(N_0, \ldots, N_5, L)(t)| \\ \lesssim\ & N_1^{-5} \prod_{j=0}^{2} \|P_{N_j} u_j(t)\|_{L^2(M)} \|P_L(P_{N_3} u_3 P_{N_4} u_4 P_{N_5} u_5)(t)\|_{L^2(M)} \end{aligned}$$

and Hölder's inequality and Lemma 3.4 imply

$$\int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim N_1^{-5} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \prod_{j=0}^{5} \|P_{N_j} u_j\|_{L^\infty(\tau_0; L^2(M))}.$$

which together with (29) gives

$$\sum_{L \geq 0} \Big| \int_{\tau} I(N_0, \dots, N_5, L)(t) dt \Big| \lesssim N_1^{-1} \prod_{j=0}^{5} \|P_{N_j} u_j\|_{V^2_\Delta}.$$

Dyadic summation easily yields

$$\Sigma_{21} \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{X^s} \prod_{j=1}^{5} \|u_j\|_{X^1}.$$

The contribution of Σ_{22}, where $N_0 \gg N_1 \geq N_2 \geq \ldots \geq N_5$ can be treated in the same way by switching the roles of N_1 and N_0. □

The proof of Theorem 4.1 – based on Proposition 4.2 and the contraction mapping principle – is standard and can be concluded as in [14, Section 4], cp. also the references therein.

Appendix A. Proof of Lemma 3.1

For the sake of completeness, we include a proof of Lemma 3.1 here. This result has been proved in the case $J = [1, N]$ and $\mu_n^2 = n^2$ in [2, Section 4] and stated for general J in the case $p = 6$ in [7]. More precisely, we describe here the necessary modifications with respect to Bourgain's original work [2, Section 4]. We closely follow the presentation in [2, Section 4]. For this reason we work in the 1-periodic (instead of the 2π-periodic) setup here. However, note that we replaced $\delta N^{\frac{1}{2}}$ with λ. We also refer the reader to [3, Section 3], and to the book [20] for more details on the circle method of Hardy and Littlewood.

First of all, we switch from t to $-t$, and reduce the estimate (3.1) for general $\alpha \in \mathbb{N}_0$ to the case $\alpha = 0$. The latter simply follows by dilating time by the factor 16 and applying the result for $\alpha = 0$ on $[0, 2\pi]$ to a modified sequence and the translated and dilated interval $4J + \alpha$. From now on we will assume that $\mu_n^2 = n^2$.

Let $\mathbb{N}$ denote the set of positive integers, and let $J = [b, b + N]$, $b, N \in \mathbb{N}$. As in [3, Section 3] we choose a sequence σ satisfying

(i) For all $n \in \mathbb{Z}$: $0 \leq \sigma_n \leq 1$; for all $n \in J$: $\sigma_n = 1$ for $n \in J$; for all n such that $n < b - N$ or $n > b + 2N$: $\sigma_n = 0$.
(ii) The sequence $(\sigma_{n+1} - \sigma_n)$ is bounded by N^{-1} and has variation bounded by N^{-1}.

Let $p > 4$, and $0 < \varepsilon \ll 1$ such that $p - \varepsilon > 4$. Our aim is to prove the distributional inequality

$$\sup_{b \in \mathbb{N}} \Big| \Big\{ t \in [0,1] : \Big| \sum_{n \in \mathbb{Z}} c_n \sigma_n e^{2\pi i t n^2} \Big| > \lambda \Big\} \Big| \leq C_\varepsilon N^{\varepsilon/2} \lambda^{-4-\varepsilon}, \tag{30}$$

for all c_n such that $\sum_n |c_n|^2 = 1$, which implies (13), because the set is empty if $\lambda \gg N^{1/2}$.

a) There exists $c_\varepsilon > 0$ such that

$$\sup_{k \in \mathbb{N}, b \in \mathbb{N}_0} \#\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq N; \quad n_1(n_2 + b) = k\} \leq c_\varepsilon N^{\frac{\varepsilon}{400}}.$$

In the case $0 \leq b \leq 10N^2$ this follows from the standard bound on the number of divisors function, see [12, Theorem 315]. Otherwise, the set contains at most one element. With this ingredient one can easily modify the argument in [2, formulas (1.3)-(1.6)] to deduce

$$\Big\| \sum_{n \in \mathbb{Z}} c_n \sigma_n e^{2\pi i t n^2} \Big\|_{L^4_t(0,1)} \lesssim N^{\frac{\varepsilon}{400}} \Big(\sum_{n \in \mathbb{Z}} |c_n|^2 \Big)^{\frac{1}{2}}. \tag{31}$$

This bound implies (30) for $\lambda \lesssim N^{1/2 - \nu/4}$ for $\nu = 1/100$, so it remains to prove (30) in the case $N^{1/2-\nu/4} \ll \lambda \leq N^{1/2}$.

b) It holds

$$\Big| \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i t n^2} \Big| \lesssim q^{-1/2} (|t - a/q| + N^{-2})^{-1/2}, \tag{32}$$

for any $1 \leq a < q < N$, $\gcd(a,q) = 1$ and $|t - a/q| < (qN)^{-1}$. The claim (32) follows from

$$\sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i t n^2} = e^{2\pi i t b^2} \sum_{m \in \mathbb{Z}} \sigma_{b+m} e^{4\pi i b t} e^{2\pi i t m^2}$$

and [3, Lemma 3.18] with $x = 2bt$. Estimate (32) replaces [2, formula (4.10)], with

$$f(t) = \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i t n^2}.$$

c) We define the major arcs $\mathcal{M}$ to be the disjoint union of the sets

$$\mathcal{M}(q,a) = \{t \in [0,1] : |t - a/q| \leq N^{\nu-2}\},$$

for any $1 \leq a \leq q \leq N^\nu$, $\gcd(a,q) = 1$. Let $t \in [0,1] \backslash \mathcal{M}$. By Dirichlet's Lemma there exists a reduced fraction a/q with $1 \leq a \leq q \leq N^{2-\nu}$ such that $|t - a/q| \leq N^{\nu-2}$, and since $t \notin \mathcal{M}$ it must be $q > N^\nu$ and (32) implies

$$|f(t)| \leq N^{1-\nu/2}, \tag{33}$$

which replaces [2, formula (4.6)].

d) In order to prove (30), it therefore suffices to prove a bound on the number R of N^{-2}-separated points $t_1, \dots, t_R \in [0,1]$ where

$$\Big| \sum_{n\in\mathbb{Z}} c_n \sigma_n e^{2\pi i n^2 t_r} \Big| > \lambda.$$

We recall that $N^{1/2-\nu/4} \ll \lambda \leq N^{1/2}$. As in [2] we obtain

$$\sum_{1\leq r,r'\leq R} \Big| \sum_{n\in\mathbb{Z}} \sigma_n e^{2\pi i n^2 (t_r - t_{r'})} \Big| > \lambda^2 R^2.$$

For fixed $\gamma > 2$ this estimate and Hölder's inequality yield

$$\sum_{1\leq r,r'\leq R} |f(t_r - t_{r'})|^\gamma > \lambda^{2\gamma} R^2,$$

which replaces [2, formula (4.13) with $\lambda = \delta N^{\frac{1}{2}}$]. From here, the arguments in [2, pp. 305–307] apply verbatim and show that

$$R \lesssim \lambda^{-4-\varepsilon} N^{2+\frac{\varepsilon}{2}}.$$

Because of the N^{-2}-separation property of the points $t_1, \dots, t_R$ this implies

$$\Big| \Big\{ t \in [0,1] : \Big| \sum_{n\in\mathbb{Z}} \sigma_n c_n e^{2\pi i t n^2} \Big| > \lambda \Big\} \Big| \lesssim N^{-2} \lambda^{-4-\varepsilon} N^{2+\frac{\varepsilon}{2}},$$

which gives (30).

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